Also solved by Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5466: *Proposed by D.M. Bătinetu– Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" School, Buzău, Romania*

Let $f : (0, +\infty) \to (0, +\infty)$ be a continuous function. Evaluate

$$\lim_{n\to\infty} \int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right) dx.$$

Solution 1 by Moti Levy, Rehovot, Israel

The mean value theorem of the integral calculus states:
Let $f(x)$ be continuous function, then

$$\int_a^b f(x)\, dx = (b-a)\, f(\xi)\,, \;\; a \le \xi \le b.$$

Therefore,

$$\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right) dx = \left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}\right) f\left(\frac{\xi}{n}\right), \quad \frac{n^2}{\sqrt[n]{n!}} \le \xi \le \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}.$$

Taking limits of both sides,

$$\lim_{n\to\infty} \int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right) dx = \lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}\right) \lim_{n\to\infty} f\left(\frac{\xi}{n}\right).$$

Since $f(x)$ is continuous then

$$\lim_{n\to\infty} f\left(\frac{\xi}{n}\right) = f\left(\lim_{n\to\infty} \frac{\xi}{n}\right) = f\left(\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}}\right)$$

Using Stirling's asymptotic formula, we have

$$\sqrt[n]{n!} \sim \frac{n}{e}. \tag{1}$$

By (1),

$$\frac{n}{\sqrt[n]{n!}} \sim e, \quad \frac{n^2}{\sqrt[n]{n!}} \sim e \cdot n,$$

which implies that

$$f\left(\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}}\right) = f(e)$$

and that

$$\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}} \sim e.$$

We conclude that

$$\lim_{n\to\infty}\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx = ef\left(e\right).$$

Solution 2 by Bruno Salgueiro Fanego,Viveiro, Spain

Let's proceed as in http://www./oei.es/historico/oim/revistaoim/numero 53/261_Bruno.pdf:
Let $n \in N$; since f is continuous on (x_n, x_{n+1}), by the mean value theorem of integral calculus, we have that $\int_{x_n}^{x^{n+1}} f\left(\frac{x}{n}\right)dx = f\left(\frac{\xi_n}{n}\right)(x_{n+1}-x_n)$for some $\xi_n \in (x_n, x_{n+1})$.
Since$\frac{x_n}{n} < \frac{\xi_n}{n} < \frac{x_{n+1}}{n}$,

$$\lim_{n\to\infty}\frac{x_n}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{n^n n!}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}} = \lim_{n\to\infty}\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty}\frac{(n+1)^n(n+1)n!}{(n+1)!n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$$

and $\lim_{n\to\infty}\frac{x_{n+1}}{n} = \lim_{n\to\infty}\frac{x_{n+1}}{n+1}\cdot\frac{n+1}{n} = \lim_{n\to\infty}\frac{x_{n+1}}{n+1}\cdot\lim_{n\to\infty}\frac{n+1}{n} = e\cdot 1 = e$, by the sandwich rule we obtain that $\lim_{n\to\infty}\frac{\xi_n}{n} = e$, and, hence,
$\lim_{n\to\infty} f\left(\frac{\xi_n}{n}\right) = \left(\lim_{n\to\infty}\frac{\xi_n}{n}\right) = f(e)$.
Moreover, from Stolz' rule,

$$\lim_{n\to\infty}(x_{n+1}-x_n) = \lim_{n\to\infty}\frac{(x_{n+1}-x_n)}{(n+1)-n} = \lim_{n\to\infty}\frac{x_n}{n} = e.$$

So, the required limit is equal to

$$\lim_{n\to\infty}\int_{x_n}^{x_{n+1}} f\left(\frac{x}{n}\right)dx = \lim_{n\to\infty} f\left(\frac{\xi_n}{n}\right)\lim_{n\to\infty}(x_{n+1}-x_n) = ef(e).$$

Solution 3 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

This particular problem is similar to Problem 121, which was proposed by D.M. Bătinetu-Giurgiu ("Matei Basarab" National College, Bucharest, Romania) and Neculai Stanciu ("George Emil Palade" School, Buzău, Romania) to the Math Problems Journal, Volume 5, Issue 2 (2015), pp. 420-421. We'll use the following lemma.
Lemma: Let $f : [a, b] \to \Re$ be continuous and $(x_n)_n, (y_n)_n$ two convergent sequences of $[a, b]$ that have the same limit c, then

$$\int_{x_n}^{y_n} f(t)dt = f(c)(y_n - x_n) + O(y_n - x_n).$$

Proof: Let $\epsilon > 0$, then there exists $\delta > 0$ such that $|f(t) - f(c)| < \epsilon$, whenever $|x - c| < \delta$. Since $x_n, y_n \to c$, the there is an $n_0 \in N$ such that $x_n, y_n \in (C - \delta, C + \delta)$, whenever $n > n_0$. Therefore,

$$\left| \int_{x_n}^{y_n} f(t)dt - f(c)(y_n - x_n) \right| \leq \int_{x_n}^{y_n} |f(t) - f(c)|\, dt \leq \epsilon\, |y_n - x_n|\,.$$

Note that the given integral equals

$$I_n = n \int_{\frac{n}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{n\sqrt[n+1]{(n+1)!}}} f(t)dt,$$

this comes directly from the substitution $t = \dfrac{x}{n}$. Let x_n, y_n be the lower, upper bound of the last integral respectively then $x_n, y_n \to e$, since $\dfrac{n}{\sqrt[n]{n!}} \to e$, and thus $\dfrac{(n+1)^2}{n\sqrt[n+1]{(n+1)!}} = \dfrac{n+1}{n}\dfrac{(n+1)}{\sqrt[n+1]{(n+1)!}} \longrightarrow e$, as $n \longrightarrow \infty$. Note that by Stolz' theorem

$$n(y_n - x_n) = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}} \longrightarrow e, \text{ as } n \longrightarrow \infty.$$

By the lemma we have

$$I_n = n\left[f(c)(y_n - x_n) + O(y_n - x_n)\right] = ef(e) + O(1),$$

which proves that the limit equals $ef(e)$.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Soumitra Mandal, Scottish Church College, Chandan -Nagar, West Bengal, India; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposers.

5467: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

In an arbitrary triangle $\triangle ABC$, let a, b, c denote the lengths of the sides, R its circumradius, and let h_a, h_b, h_c respectively, denote the lengths of the corresponding altitudes. Prove the inequality

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a \cdot h_b \cdot h_c}},$$

and give the conditions under which equality holds.

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

We know that $h_a = (bc)/(2R)$ and cyclic so the inequality actually is

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq \frac{3abc}{2R}\left(\frac{8R^3}{(abc)^2}\right)^{\frac{1}{3}} = 3(abc)^{\frac{1}{3}}.$$

We prove the stronger one